The Theory of Anomalous Scale Dimensions

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Abstract

Using the previously gained insight about the particle/field relation in conformal quantum field theories which required interactions to be related to the existence of particle-like states (infraparticles) associated with fields which necessarily have an anomalous contribution in addition to their canonical scale-dimensions, we set out to construct a classification theory for the spectra of anomalous dimensions. We find that they are resulting from a braid group structure. The latter is however not related to statistics (spacelike interchange) but rather draws its raison d'etre from the timelike Huygens principle (timelike commutativity), a characteristic property of conformal observables, as well as from the existence of a timelike ordering. The global aspects of this Huygens structure also leads to a timelike global charge transport around the Dirac-Weyl compactified Minkowski world $\bar{M}$ which in turn is inexorably related with an S-T modular SL(2,Z) group structure.

1 Background and preview of new results

It had been known for a long time that conformal quantum field theory exhibits in addition to the general spin-statistics theorem another more characteristic structural property which we will refer to as the "anomalous dimension-central phase" connection. It relates the anomalous scale dimension of fields modulo integers (semi-integers in the case of Fermion fields) to the phase obtained by performing one complete timelike sweep around the compactified Minkowski world [1] and hence is analogous to the relation of the spin value related to a spatial rotation sweep to the statistics phase [2] of the spin-statistics connection including chiral conformal field theory where the rotation around S^1 is lightlike. The word "central" here refers to the center $Z(\widetilde{SO(d,2)})$ of the infinite sheeted covering group $\widetilde{SO(d,2)}$ which has one abelian generator for spacetime dimensions $d > 2$. It is our aim to show that behind this connection there is a classification theory which determines the possible values which the spectrum of the anomalous scale dimensions can take in terms of a timelike superselection structure. To be more specific, the Huygens principle which holds for timelike separated conformal observables and demands that they commute, together with the existence of a timelike ordering structure generates a situation which resembles that of spacelike observables in d=1+1 theories, including the appearance of the braid group. The difference is that the Huygens version of the braid group has no interpretation in terms of particle/field statistics which in higher dimensions remains Boson/Fermion statistics in accordance with the spin-statistics theorem. This "Huygens exchange" is best physically characterized by the consequences it leads to (e.g. imposing a structure on the anomalous dimension spectrum); if one wants to interprete it literally its is a somewhat strange exchange where an object of compact spatial extend and finite duration and another similar object which appears in the same region as the first but at a later time after the first has gone become interchanged and the two states are compared. So if one resolves this imagined process in time, it is something as

$$local\,vacuum \rightarrow object\,I \rightarrow local\,vacuum \rightarrow objectII \rightarrow local\,vacuum \qquad (1)$$

as compared to I and II interchanged.

In d=1+1 dimensions one has accumulated a good understanding of conformal theories and their associated superselected charge structure. One knows that they can be decomposed into the $x_\pm$ light cone components called chiral theories. There is a systematic way to classify localizable representation of chiral observable algebras and one finds charge-carrying fields which obey a lightlike exchange algebra in which those new objects satisfy group commutation relations either of the abelian kind (anyonic) or with nonabelian R-matrices (plektonic) with higher multiplicities and quantized statistical phases. Since the latter determine the spectrum of anomalous dimensions (=critical indices on the side of statistical mechanics) modulo integers, one has a theory of anomalous dimension as soon as one knows how to classify physically admissable representations of the infinite braid group or more precisely the ribbon braid group RB_∞. The classification of its physically admissable representations is done by the method of tracial states on B_∞ which follow a combinatorial version of the field theoretic cluster decomposition property, the so-called Markov property. This method was originally invented by DHR in order to classify the admissable permutation group statistics which is associated with the algebraic superselection theory of compactly localized charges in d¿2 [3]. The mathematicians studying subfactor theory [4] independently discovered a more general version of this method and called it very appropriately the method of "Markov traces". The name Markov in this context reveals a lot about the conceptual scope of this theory because Markov junior refers to the Russian mathematician who made important contribution to the early study of the braid group but at the same time one is invited to think about Markov senior the probabilist since for a physicist the tracial state retains a discrete version of the field theoretic cluster decomposition property[1] which for a mathematician suggests a probabilistic interpretation. The physicist reader finds a presentation which unites both methods e.g. in [5] and in an appendix of the present paper.

Although it is mathematically understood and well-known, it remained always somewhat of a conceptual miracle, that a theory of critical indices for critical phenomena in d=1+1 classical statistical mechanics draws its computational strength from the noncommutative real time side. This should in particular be a bit surprising to anybody with a thoughtful mind who has learned QFT from say post 1970 standard textbooks, since it goes opposite to the prevalent euclidean doctrine which states that best formulation+understanding+calculation method of physical (noncommutative) real time QFT goes through (formally commutative) euclidean functional integrals. Nothing of this structural richness, in particular the concepts exposed in this article, is visible if one would be limited to euclidean methods.

Anomalous dimensions are also expected in higher dimensional conformal QFT but it is as yet not known which basic physical concept is behind their structure. Certainly it is not statistics as in the d=1+1 case because it is well-known that in higher dimensional QFTs, whether conformal or not, compact localizability of charges always results in Bose/Fermi statistics. Even if one does not impose any a priori localization requirements on the superselected charges and instead postulates the existence of isolated mass-hyperboloids (mass-gap hypothesis) which is known to yield the more general semi-infinite spacelike string-like (more precisely: spacelike conic) localization associated with topological[2] charges, in $d \geq 1+3$ one still ends up with Bosons and Fermions. In d¿2 conformal theories, although the mass-gap prerequisite is not met, the statistics still remains conventional [6]. The reason is that a noncompact region e.g. a wedge ($|x_0| < x_1$, $x_\perp$ arbitrary) is conformally equivalent to a compact region (wedge→double cone); so even if one would allow objects which spread over a wedge, one would still end with conventional statistics. which is the localization used in the DHR theory [3]. In this case one knows that the Dirac-Weyl compactified Minkowski space $\bar{M}$ is the right arena for the observables which fulfill the Huygens principle i.e. they are not only commutative for spacelike distances but continue to be so for timelike separations so that loosely speaking all interactions propagate on the light cone.

Let us for a moment return to d=1+1 conformal theories. As it is well known they factorize into their two chiral parts with a classifiable braid group structure on each (compactified) light ray S^1. In fact the physical concepts also supply

[1] The all-pervading cluster property in its simplest quantum mechanical version states that doing (n+1)-particle physics and forcing one particle to be a spectator (by shifting it towards infinity) one must recover the previously studied n-particle quantum physics. The only group which has this inclusive Russian matrushka structure is B_∞ and as a special case S_∞.

[2] The reader is cautioned against any premature reading of words which he may have met in connection with the geometry underlying euclidean functional integrals. Here we are talking about real time local quantum physics where "topological charge" has a direct physical definition and interpretation in terms of localization [6].

a Markov trace on the infinite braid group B_∞ whose use in a GNS construction yields a representation of the ribbon (twisted) braid group. In addition the existence of lightlike covering transformation via the global aspect of the Moebius group (conformal rotation) together with the construction of so-called global charge transporters around each compact world S^1 provide a so-called S-T generating system for the modular group SL(2,Z)[3] [9][10]. The Markov trace formalism on the braid group together with the formalism of global charge transport leads to an extended Markov formalism which includes the mapping class group for all genii and 3-manifold invariants, but these objects here play a more abstract role in the algebraic local quantum physics aspects and are not related to the living space in the sense of localization of QFT i.e. we are not dealing with QFT on Riemann surfaces or 3-manifolds rather this richness arises from conformally globalized lightray spacetime. This combinatorial theory of intertwiners (often called topological field theory) is in fact the best extraction of a discrete structure resembling e.g. the group representation theory in Wigner's approach to symmetry in QT. The fact that SL(2,Z) does not act on objects like a Wigner symmetry group is however a reminder that in this situation inner and spacetime symmetries are inexorably related (see previous footnote) and that a complete separation is impossible. The Markov traces on B_∞ allow a rather systematic classification (see last section) and the theory of anomalous dimensions modulo integer values is completely determined by the classifiable combinatorial structure. One even knows how to construct the associated chiral QFTs from these data. It is also known how by a tensor product construction we may obtain d=1+1 local fields. Different from the chiral observables, these local fields (example the order field in the conformal Ising model whose commutator does not vanish for timelike separations) do not obey the Huygens principle. In fact the chiral observable algebras consists precisely those operators which obey Huygens principle $\mathcal{A}_{Huy} = \mathcal{A}_{chir}^{(+)} \vee \mathcal{A}_{chir}^{(-)} \equiv A_{chir}^{total}$ and therefore we have a genuine inclusion of the "Huygens net" into the local (Einstein causal) net

$$\mathcal{A}_{Huy}(\bar{M}) \subset \mathcal{A}(\tilde{M}) \subset \mathcal{F}(\tilde{M}) \tag{2}$$

where we included the information of the natural "living" spaces; $\bar{M}$ stands for the Dirac-Weyl compactified (two-dimensional) Minkowski space and $\tilde{M}$ is its universal covering. $\mathcal{F}(\tilde{M})$ stands for the net of all plektonic ($\pi\lambda\varepsilon\kappa\tau\omega\sigma \simeq$ braided) fields whereas $\mathcal{A}(\tilde{M})$ denotes the local (Einstein-causal) subnet.

In higher dimensions the chiral doubling of degrees of freedom is absent and all fields are local. In that case a natural dichotomy between observables and charge-carrying operators is defined by requiring that they are those objects on M which allow a natural extension on $\bar{M}$ (in a classical analogy functions which thanks to certain fall-off properties at infinity are extendible to $\bar{M}$). Since

[3] This modular group plays an active role as a kind of internal/external "hybrid" symmetry group on Gibbs thermal states formed from the "rotational" generator, a situation which is outside the assumptions of the No-Go theorem for nontrivial (non tensor product) interplay between internal with spacetime symmetries.

these are precisely the ones obeying Huygens principle we have a natural higher dimensional analogy to A_{chir}^{total}. All operators which are not living in $\bar{M}$ are in $\mathcal{F}(\tilde{M})$ which consists of local operators which violate Huygens principle because they need the covering space $\tilde{M}$ to live on (classical analogy: sections on a bundle with basis the connected but not simply connected $\bar{M}$). There are no intermediate extensions as in (2) since the compensation formalism between tensor product factors is absent here. Of course one may always construct observables by fusing multi-local products of localized charged operators in $\mathcal{F}(\tilde{M})$ but this will not extend $\mathcal{A}_{Huy}(\bar{M})$.

We will show in this paper that as a result of the presence of the timelike Huygens property for observables and the existence of a global timelike ordering (global causality) we will have charge-carrying fields on $\tilde{M}$ which fulfill a plektonic timelike exchange algebra and lead to a S-T SL(2,Z) modular structure. Contrary to massive theories where the timelike region which is the arena of dynamics unfortunately has remained impenetrable against any attempt of a structural analysis, the local quantum version of Huygens principle for conformal observables partially "kinematizes" the structure of anomalous dimensional fields[4]. It reintroduces the above braid group structure of chiral conformal QFT but now with a very different physical interpretation which is not related to particle or field statistics. This loss of a statistics interpretation is more than compensated by the consolation of a completely new realm of a genuine spacetime/internal symmetry marriage as never seen before in QFT. We are referring here to an extremely surprising way of evading the Coleman-Mandula prohibition against such marriages (nontrivial union of spacetime- with internal symmetries) in higher dimensional massive QFT with a particle (including scattering theory) interpretation. Whereas supersymmetry only made a little almost kinematical dent (graded groups were not taken into account in the assumptions made by Coleman and Mandula), the present structure leads to a dynamical maximal sin of trespassing: all the superselection sectors which appear in the decomposition theory and even the multiplicities (in case of plektonic realizations of the braid group) of these 4-dim. conformal QFT are of spacetime origin! Of course on can always introduce additional internal symmetries by hand, but in view of the fact that such ad hoc procedure go against the spirit of viewing reality as the unfolding of physical principles and that exact nonabelian internal Wigner symmetries of the standard kind have never been seen in Nature, this is not a very attractive possibility.

In a previous letter, we have given a physical motivation why conformal QFT and in particular a theory of anomalous scaling dimensions may be important for particle physics [11]. Since, as was shown there, conformal interactions are inconsistent with a particle structure, one is required to study particle-like objects which are created by anomalous dimension fields.

The content is organized as follows. In the next section we present some geometrical prerequisites which are useful for setting up the stage for the global

[4] The role of the net of observable algebras as a "shadow" of the full theory in the reconstruction of the latter is similar to Marc Kac's problem "How to hear the shape of a drum".

conformal (block) decomposition theory in the third section. The BPZ conformal block decomposition theory [12] is too special since it makes heavy use of chiral algebras which are not available in higher dimensions. Fortunately there exists a much older little noticed decompositions theory [1] (written too early as it seems) with respect to the center of the conformal covering group of which the BPZ theory is a special case and which works also in higher dimensions . In the fourth section we will present this decomposition theory which then will be used in section 5 for the formulation of the timelike braid group structure for its centrally irreducible components. The considerations are on the level of consistency arguments. Only in the last section we finally utilize the mathematical rigor and the conceptual tightness of algebraic QFT to secure the timelike exchange algebra and modular S-T structure of our theory of anomalous dimensions. In fact it is formally a refinement of the DHR theory which is only possible in the conformal setting

The appendix contains an exposition of the DHR-Wenzl projector method for the classification of Markov-trace braid group representations. Although in this DHR-like form it has entered the physics literature [5], it remained widespread unknown among physicists. I feel that this pretty method deserves more attention and therefore I give a self-contained presentation instead of just referring to the original paper.

The reader familiar with the ongoing discussion about the structure of conformal supersymmetric Yang-Mills (SYM) theories may be curious about how the present work relates to those problems. Perturbative supersymmetry generally has a tendency to drive theories towards free fields in the sense that it perturbatively "protects" many objects against the influence of interactions. A conformal QFT is particularly sensitive against such protection; if as a consequence of this mechanism there exists one field with a canonical dimension, then this field is necessarily a free field and the sector it creates has no interaction [11]. The observable fields on $\bar{M}$ with higher integer/semiinteger dimensions should have correlation functions which stay away from those of composites of free fields. It is not known what a weak protection as e.g. the proportionality of the 2- and 3-point normalization constants to those of associated composite of free fields leads to; from experience with perturbation theory one would find it unlikely that something like this can happen in a genuinely interacting theory but intuition is not a good guide in such matters. The interesting question is where is the borderline between interacting and free observables, what means "too much"protection. The AdS-CQFT correspondence which started with the observation of a relation between two different Lagrangian theories of which one is the already mentioned SYM theory [13] was later shown to be an isomorphism (a correspondence in both directions, i.e. including the AdS bulk) in the sense of the (field coordinatization independent, intrinsic) algebraic QFT [14]. This has the consequence that if one side is Lagrangian the other side e.g. the conformal side has necessarily additional degrees of freedom which are not compatible with pointlike fields underlying a Lagrangian description. Since the statement about the isomorphism and its action is a rigorous structural theorem on the level of TCP or spin&statistics, there is a serious problem with the string induced idea of a *Lagrangian* relation, a fact which has been already pointed out before [15].

Before we enter the details of the presentation I suggest that the reader unaccustomed with methods of AQFT should skip upon first reading the last section. The meaning of many statements (including their consistency) in this paper can be understood before one looks at their proofs.

2 Covering space and decomposition theory

The consideration of the previous section has shown that the physically interesting interacting fields in a conformal theory are those with anomalous dimension. Hence it is of interest to know the spacetime interpretation of such fields. If one looks at the two-point functions of fields with integer versus noninteger anomalous dimensions one notices that their structure in the anomalous case is not compatible with a Minkowski space localization [11]. For aspects of global localization interacting conformal field theories require the introduction of the covering of (conformally compactified) Minkowski space. This is a well-studied old subject [17][1][18], especially but not only in d+1+1 [12] where conforms observables decompose additively in to the two light cone components which act on a tensor-factorized Hilbert space.

The fastest way to motivate the physical interest in conformal theories and to obtain the formalism and physical use of the conformal covering space is to notice that the Wigner representation theory for the Poincaré group for zero mass particles allows an extension to the conformal symmetry: Poincaré group$(d) \rightarrow SO(d,2)$. Besides scale transformations, this larger symmetry also incorporates the fractional transformations (proper conformal transformations)

$$x' = x - bx^2 1 - 2bx + b^2x^2 \tag{3}$$

It is often convenient to view this formula as the translation group transformed with the hyperbolic inversion

$$x \rightarrow -xx^2 \tag{4}$$

acting as an equivalence transformation within an extended group. For fixed x and small b the formula (3) is well defined, but globally it mixes finite spacetime points with infinity and hence requires a more precise definition in particular in view of the positivity energy-momentum spectral properties in its action on quantum fields. Hence as preparatory step for the quantum field theory concepts one has to achieve a geometric compactification. This starts most conveniently from a linear representation of the conformal group SO(d,2) in 6-dimensional auxiliary space $\mathbb{R}^{(d,2)}$ (i.e. without field theoretic significance) with two negative (time-like) signatures

(5)

$$G = \begin{pmatrix} g_{\mu\nu} & & \\ & -1 & \\ & & +1 \end{pmatrix}$$

and restricts this representation to the (d+1)-dimensional forward light cone

$$LC^{(d,2)} = \{\xi = (\xi, \xi_4, \xi_5\}\,; \xi^2 + \xi_d^2 - \xi_{d+1}^2 = 0\} \tag{6}$$

where $\xi^2 = \xi_0^2 - \vec{\xi}^2$ denotes the d-dimensional Minkowski length square. The compactified Minkowski space is obtained by adopting a projective point of view (stereographic projection)

$$M_c^{(d-1,1)} = \left\{ x = \xi\xi_d + \xi_{d+1}; \xi \in LC^{(d,2)} \right\} \tag{7}$$

It is then easy to verify that the linear transformation which keep the last two components invariant consist of the Lorentz group and those transformations which only transform the last two coordinates yield the scaling formula

$$\xi_d \pm \xi_{d+1} \to e^{\pm s}(\xi_d \pm \xi_{d+1}) \tag{8}$$

leading to $x \to \lambda x, \lambda = e^s$. The remaining transformations, namely the translations and the fractional proper conformal transformations, are obtained by composing rotations in the ξ_i-ξ_d and boosts in the ξ_i-ξ_{d+1} planes.

The so obtained spacetime is most suitably parametrized in terms of a "conformal time" τ

$$\begin{aligned} M_c^{(d-1,1)} &= (sin\tau, \mathbf{e}, cos\tau),\ e \in S^3 \\ t &= sin\tau e^d + cos\tau,\ \vec{x} = \vec{e}e^d + cos\tau \\ e^d + cos\tau &> 0,\ -\pi < \tau < +\pi \end{aligned}$$

so that the conformally compactified Minkowski space is a piece of a multi-dimensional cylinder carved out between two d-1 dimensional boundaries which lie symmetrically around $\tau = 0, \mathbf{e} = (\mathbf{0}, e^d = -1)$ where they touch each other [18]; but the projective aspect of Hilbert space vectors as representing physical states demands that we use the universal covering space which is the full cylinder (which has a tiling into infinitely many ordinary Minkowski spaces, Fig.1)

$$\widetilde{M_c^{(d-1,1)}} = S^{d-1} \times \mathbb{R} \tag{9}$$

Indeed in order not to be limited by the narrow confines of Huygen's principle (which tends to limit conformal relativistic system to non-interacting ones [20]), the "nature" of local quantum physics demands the use of the covering space (or as the substitute a conformal decomposition theory of local fields into irreducible components with respect to the center of the conformal covering) as will become clear in the next section. The relevance of this covering space for the notion of relativistic causality was first pointed out first by I. Segal [17] and the above parametrization which became standard in conformal QFT appears the work of Luescher and Mack [18].

Formally it solves the "Einstein causality paradox of conformal quantum field theory" [7] which originated from "would be" conformal models (infinitesimally conformal invariant) of quantum field theory as the massless Thirring model which violates Huygens principle. The naive reason for this apparent violation was that there exist continuous curves of conformal transformations which lead from spacelike separations with one point at the origin via the lightlike infinity to timelike separation which obviously generates a contradiction with the locality structure of the Thirring model whose timelike anti-commutator unlike the spacelike one does not vanish. The covering structure formally solves this causality paradox by emphasizing that the path through lightlike infinity was in fact a path which led into another copy (sheet) of a Minkowski world and it is only the unjustified projection of one of the end-point back into the compact Minkowski space region (**??**) which has a timelike distance and not the point itself (which remains causally disjoint). If one depicts the covering space as a cylinder (Fig.1), then it contains infinitely many copies of the original Minkowski space which appear in the projection to $(\xi_2, ..\xi_{d-1}) = (0, ..0)$ subspace as a finite rhomboid region [18].

Using the above parametrization in terms of $\mathbf{e}$ and the "conformal time" τ, one can immediately globalize the notion of time like distance and one finds the following causality structure ([17][18])

$$\begin{aligned} &(\xi(\mathbf{e}, \tau) - \xi(\mathbf{e}', \tau'))^2 > 0, \ \mathit{hence} \\ &\tau - \tau' > 2Arcsin\,(\mathbf{e} - \mathbf{e}'4)^{12} = Arccos\,(\mathbf{e} \cdot \mathbf{e}') \end{aligned}$$

Since it is expressed in terms of the difference of two light cone coordinates, a conformal transformation which is linear in the ξ-variables leaves it invariant. For the description of the Dirac-Weyl compactified Minkowski space the use of the following simpler parametrization is more convenient

$$\begin{aligned} &\xi^\mu = x^\mu \\ &\xi^4 = 12(1 + x^2) \\ &\xi^5 = 12(1 - x^2) \\ &i.e. \ \ (\xi - \xi')^2 = (x - x')^2 \end{aligned} \tag{10}$$

The formulation in terms of conformal covering space would be useful if the world (including laboratories of experimentalists) would also be conformal, which certainly is not the case. Therefore it is helpful to know that there is a way of re-phrasing the physical content of local fields (which violate the Huygens principle and instead exhibit the phenomenon of "reverberation" [7] inside the forward light cone) in the Minkowski world of ordinary particle physics[5] without running into the trap of the causality paradox of the previous section; in this way the use of the above explicit parametrization would loose some of its importance. This was first achieved in a joint paper involving one of the present authors [1] whose main point was that the global causality structure could be taken care of in terms of a global decomposition theory of fields with respect to the center of the conformal covering (conformal block decomposition). Local fields, although behaving irreducibly under infinitesimal conformal transformations, transform in general reducibly under the action of the global center of the covering $Z(\widetilde{SO(d,2)})$. As a unitary abelian group it is generated by the 2π-translation in the conformal time τ. A local covariant field $A(x)$ (local in the sense of the causal structure of $\widetilde{SO(d,2)}$) corresponding unitary operator $Z \in Z(\widetilde{SO(d,2)})$ can be decomposed as [1]

$$A_d(x) = \int_0^! A_d^\xi(x) d\xi \tag{11}$$

with A_d^ξ formally given by

$$A_d^\xi(x) = \sum_{n=-\infty}^{\infty} Z^n A_d(x) Z^{-n} exp\left[in\pi(d-2\xi)\right] \tag{12}$$

from which one gets

$$AdZ A_d^\xi(x) = exp\left[-i\pi(d-2\xi)\right] A_d^\xi(x) \tag{13}$$

The notation is the following: d is the scaling dimension of the local (causal in the covering sense) field $A_d(x)$ and the ξ-integration is the decomposition into its centrally irreducible components. These component fields, unlike the original globally causal fields, do not fulfill the Reeh-Schlieder theorem (sometimes referred to as the field—field-state-vector correspondence), rather they have a source and a range and their application to a non-matching source subspace vanishes. Their physical interpretation is easily obtained from the conformal analysis of 3-point functions

$$\langle C_{dc}(x) A_d(y) B_{d_b}(z)\rangle = \left\langle C_{dc}(x) A_d^\xi(y) B_{d_b}(z)\right\rangle$$
$$\xi = 12(d + d_b - d_c) \mod(1)$$

[5] As with spinor fields where a complete spatial rotation produces a minus sign, the complete timelike "rotation" once around the compactified Minkowski space will produce phase factors on the central components which are generally different for different components of the original globally causal field.

Hence the quantum number of the irreducible components is related to the dimensional spectrum (critical indices) of the theory and the ξ-dependent phase factors enter the transformation law which comes close to the naive classical transformation

$$U(b)A_d^{\xi}(x)U^{-1}(b) = 1\left[\sigma_+(b,x)\right]^{d-\xi}\left[\sigma_-(b,x)\right]^{\xi}A_d^{\xi}(x) \tag{14}$$

whereas the more complicated law for the local field follows from the decomposition formula (11). Structural properties of the real time formulation as this remain totally hidden in the euclidean formulation.

In the case of d=1+1 for which the group (as well as its center) factorizes $\widetilde{S(2,2)} = \widetilde{SU(1,1)} \times SU(1,1)$ and one obtains the well-known BPZ [12] conformal block decomposition theory which results from the above general decomposition theory by factorization into the two light ray components (chiral decomposition). In order to facilitate the reading of the mentioned 74/75 papers on the subject [1], we have used exactly the same normalizations and notation. There is a special aspect of this chiral decomposition theory. It is the only case for which one has a classification theory of the possible spectra of dimensions/critical indices. It is given by the spectrum of phases of the so-called statistics parameter which occurs within the superselection theory of AQFT, and in the case at hand is inexorably related to the timelike braid group exchange algebra structure of the nonlocal irreducible components A_d^{ξ}.

The structure of the center in chiral conformal field theories is determined by the discrete spectrum of the rotation operators for the compactified $\pm$lightrays $R^{(\pm)} = L_0^{(\pm)}$ where the last notation is the one used in the approach to chiral theory based on the Virasoro algebra decomposition of the energy-momentum tensor. It is well-known that this operator shares with the light ray translations $P^{(\pm)}$ the positivity of its spectrum. This becomes in fact obvious if one represents it in terms of P

$$\begin{aligned} R^{(\pm)} &= P^{(\pm)} + K^{(\pm)} \\ K^{(\pm)} &= I^{(\pm)}P^{(\pm)}I^{(\pm)} \end{aligned} \tag{15}$$

where $I_{\pm}$ is the representer of the chiral conformal reflection $x \rightarrow -1x$ (in linear lightray coordinates x) and K is the generator of the fractional special conformal transformation (3). However the two-dimensional inversion (4) does not factorize since the chiral inversion rewritten in terms of vector notation corresponds to

$$\begin{aligned} x_0 &\rightarrow -x_0x^2 \\ x_1 &\rightarrow x_1x^2 \end{aligned} \tag{16}$$

The "wrong" sign in the spacial part can be corrected by a parity transformation $x_+ \leftrightarrow x_-$ which mixes the two chiral components. In defining an object which transforms as a vector this has to be taken in consideration

$$R_\mu = P_\mu + IK_\mu I \tag{17}$$
$$I = Parity \cdot (I_+ + I_-) \tag{18}$$

The full center is generated by the finite rotations $e^{iR^{(\pm)}\tau}$ at $\tau = 2\pi$. In order to find those formulas which generalize to higher dimensions, we should restrict the covering group actions to local fields. On bosonic spaces the center is generated by just one central element $e^{i(R^{(+)}+R^{(-)})2\pi} = e^{iR^{(0)}2\pi}$ as a result of the identity $e^{i(R^{(+)}-R^{(-)})2\pi} = 1$ which holds on the smaller space generated cyclically from the vacuum by the application of d=1+1 Bose fields. The generators of the center are often called the (abstract) T-transformations and in the chiral theory there exist charge transporters around the circle which give rise to an (abstract) Verlinde matrix S. Both matrices together form the modular SL(2,Z) group. Note that the T-part is of a more spacetime origin whereas the properties of charges are conceptually closer to internal symmetries. The modular group SL(2,Z) combines both aspects and it comes as no surprise that this modular group is not a Wigner symmetry group of quantum theory. In fact S does on ground state representations fulfilling the spectrum condition but rather acts on the analytic continuation of thermal correlation functions which result from chiral Gibbs states associated with the rotation "Hamiltonian".

The vector formula (17) is valid in any dimension i.e. does not require light ray factorization. It leads to a family of operators with discrete spectrum $e \cdot R$ which are dependent on a timelike vector e_μ. With them one can form higher dimensional Gibbs states of the form

$$\langle A \rangle_\beta \equiv tr(e^{-\beta e\cdot R} A) \tag{19}$$
$$A \in \mathcal{A} \tag{20}$$

which are transformed into each other by Lorentz-transformations. These are the analogues of the chiral Gibbs states associated to $e^{-\beta L_0^{(\pm)}}$. One may ask the question of how far this analogy goes; in particular whether there are S-T operations and an associated action of the SL(2,Z) modular group on R_0-thermal states.

To understand the geometric action of $e^{ie\cdot R\tau}$, it is hepful to depict the covering world $\tilde{M}$ with a copy of the Minkowski world inside. From (**??**) one obtains the identification of the covering world with the surface of a d+1 dimensional cylinder [18]. In Fig.1 only two of the d-2 components of the d-dimensional **e**-vector have been drawn, the others have been set zero. For depicting the spacelike complement of a double cone $\mathcal{O}$ in in $\tilde{M}$ it is more convenient to cut open the cylinder in τ-direction and identify opposite sides as in Fig.2.

On the other hand the living space of the observable algebra is the Dirac-Weyl compactification $\bar{M}$ of M which is depicted as Fig.3 with opposite two sides a and b identified. Vice versa the Minkowski space results from cutting the Dirac-Weyl compactification along a d-1 dimensional subspace ξ which generates a and b. Note that as a result of this identification the union of the timelike and spacelike complements form a connected set in $\bar{M}$. The first use of these geometrical properties in the setting of algebraic QFT is due to Hislop and Longo [19]

The above analogies between the d+1=1 case and the higher dimensional conformal field theory should however not lead one into overlooking a remarkable difference. Already on a purely classical level the characteric value problem for the free wave equation is totally different from either its massive counterpart or from the d¿2 conformal case. Whereas in the latter cases the data on one lightray or lightfront is complete, the zero mass d=1+1 case needs both the lightray data in order to determine the d=1+1 theory. In the QFT the manifestation of this is the tensor factorization into the chiral degrees of freedom which amounts to a doubling of degrees of freedom. In the next section we will see that this also leads to an exceptional behavior in the timelike Huygens structure and the associated timelike braidgroup structure. So the chirally factorizing d=1+1 situation is a guide in certain higher dimensionl aspects and stands in interesting contrast to others.

Sorry, as an inhabitant of the electronic bronze-age I have not been able to produce a ps-file from the 3 latexcad drawings. I will try again with some help in a second version which should be available on this weekend.

3 Central decomposition and braid group structure

After having understood that the phases appearing in the central decomposition of local fields into irreducible components (conformal block decomposition) are exponentially related to the anomalous dimensions, there remains the important question what determines the spectrum of anomalous dimensions of a conformal model or what determines these anomalous phases. Again the peculiarities of the d=1+1 timelike structure for local fields lead to an exception. Let us consider this special case first and then contrast this with the timelike exchange structure for the higher dimensional conformal theories.

It should be clear that the standard presentation of chiral conformal theory in terms of representation theory of special algebras (Virasoro-, current-, loop group etc.) is not suitable from either a physical or mathematical point of view since they have no place in higher-dimensional conformal theories and are even in d=1+1 too special for a general classification based on physical principles. Therefore it is helpful to at least sketch another method in which chiral theories really serve as a theoretical laboratory for higher dimensional ones and there is no danger of chiral sectarianism. The result of this alternative method is the characterization of chiral field theories in terms of so called exchange algebras [21] which are generated by fields fulfilling the following relation

$$
\begin{aligned}
P_{\beta_0} a_{\alpha_1}(x_1) P_{\beta_1} a_{\alpha_2}(x_2) P_{\beta_2} &= \sum_{\beta_1} \left[R^{(\beta_0\beta_2)}_{(\alpha_1\alpha_2)} \right]^{\pm 1}_{\beta_1\beta_1'} P_{\beta_0} a_{\alpha_2}(x_2) P_{\beta_1'} a_{\alpha_1}(x_1) P_{\beta_2} \qquad (21) \\
for\ x_1 &\gtrless x_2
\end{aligned}
$$

The notation is the following, the $a_a(x)$ are the charge-carrying chiral fields which live on one of the light rays i.e. x stands for either x_+ or x_-. The chiral Hilbert space in which they act is a direct sum of charged spaces $H_{chir} = \oplus_\alpha H_\alpha$ where the a_α applied to the vacuum is a special vector in H_α from which all vectors of H_α may be obtained by applying operators from the observable algebra and forming the closure. The P_α are projectors on the subspace H_α which are central with respect to the observable algebra, the letter being a bosonic/fermionic subalgebra of (21) i.e. a subalgebra generated by all operators with $R^{\cdot\cdot}_{\cdot\cdot} = \pm 1$. The best way to think about these exchange algebras is to view the notation a_α as a shorthand for the collection $[a_\alpha]_{\beta\gamma} \equiv P_\beta a_\alpha P_\gamma$. In distinction to standard (Lagrangian-, Wightman-) fields they come with a central source- and range- projector. This structure was first extracted from the n-point functions the chiral Ising model [23] (by using an idea due to Kadanoff) and later derived in a model-independent rigorous way within the framework of algebraic QFT [5] (see also remarks in next section) where it was shown that the above projected operators (or rather a bounded version ot them) are identical to the operators of the so-called “reduced field bundle” of the Doplicher-Haag-Roberts theory.

¿From the chiral exchange fields one may construct the bosonic/fermionic d=1+1 local fields. In the bosonic case they have the form (the bar denotes the conjugate charges)

$$
\psi_a(x) = \sum_{\beta\gamma} \left[a_\alpha^{(+)} \right]_{\beta\gamma} (x_+) \otimes \left[a_\alpha^{(-)} \right]_{\bar\beta\bar\gamma} (x_-) \qquad (22)
$$

and the causal commutation comes about because the natural ordering in the 2-dim. spacelike region in terms of light ray components natural ordering of the x_+ and the opposite ordering for the x_-, so that the Rs coming from the exchange of the + lightray algebra compensate with the $R^{-1}s$ from the x_- algebra. However in the timelike region the Rs amplify so that these fields do not obey Huygens principle and therefore have a nontrivial covering structure. In fact the commutation between two diagonal tensor components (22) produces according to (21) nondiagonal intermediate components whose presence prevent the validity of timelike commutation relations which only involve diagonal tensor products appearing as in (22).

In the higher dimensional case there is of course no tensor product factorization into $\pm$. Since the timelike region admits an ordering structure (which was already used in the global causality formulation), the most general commutation structure allowed by the topology of the situation is that of a timelike exchange algebra with braid group statistics for the charge carrying fields. These fields fulfill the Huygens principle with respect to the observables (which is necessary to assign a notion of localization to them) and as in chiral theories the commutation relation between them are fixed by the structure of observable algebras and can be computed. We will show in the last section that the structure of timelike observables and their localized endomorphisms permit the braid group structure, however such a structure analysis cannot answer the existence of non-trivial conformal theories. Since in higher dimensions there are no illustrations for theories with anomalous dimensions among free field theories and there are also no concrete well-defined observable algebras over which one can construct anomalous representations in addition to the vacuum representation[6]. So the possibility of constructions requires to solve this problem first for chiral algebras i.e. construct representations of exchange algebras. Although we will have some more comments on this existence problem, we will postpone explicit constructions to a separate work because the methods are different from the ones used in the present structure analysis.

The comparision ofr the time-like braid group commutation relations with the decomposition theory (12) shows that former ξ-components of the central composition are identical to the source-range fields of the exchange algebra and the phase can be read off from the braid group representation [21]

$$\begin{aligned} e^{2\pi i(d_\beta - d_{\alpha_1} - d_{\alpha_2})} &= \left[R^{(\beta 0)}_{(\alpha_1\alpha_2)} R^{(\beta 0)}_{(\alpha_2\alpha_1)}\right]^2, \ d = 1+1 \\ e^{2\pi i(d_\beta - d_{\alpha_1} - d_{\alpha_2})} &= R^{(\beta 0)}_{\alpha_1\alpha_2} R^{(\beta 0)}_{(\alpha_2\alpha_1)}, \ d > 1+1 \end{aligned} \tag{23}$$

where the $d_\alpha s$ are the either the scaling dimensions of the three fields appearing in the nonvanishing 3-point function (**??**) or the dimension of the lowest R_0 energy state in the sector α; since integers drop out in the exponent, there is no difference. It is easy to see that the special Rs which start with the vacuum projector are one dimensional monodromy factors i.e. phase factors. The square in the first formula is a reflection of the fact that for d=1+1 one is dealing with a ±tensor product structure (the d refer to the scale dimensions of the two-dimensional fields). In higher dimensions the allowed commutation relation (allowed by the interplay between timelike ordering and Huygens principle) has the desired form (21)

$$\begin{aligned} [\psi(x)]_{\beta\beta_1} [\psi(y)]_{\beta_1\beta_2} &= \sum_{\beta_1'} R_{\beta_1\beta_1'} [\psi(y)]_{\beta\beta_1'} [\psi(x)]_{\beta_1'\beta_2} \\ if \quad (x-y)^2 &> 0 \end{aligned} \tag{24}$$

i.e. the central components satisfy a timelike braid group exchange algebra relation. On the other hand the validity of say bosonic commutation relations for

[6]The authors in [22] who claim that Murphy's law which says that "everything which can go wrong will go wrong" has universal validity in physics are not entirely correct. They overlooked that in QFT the opposite holds in the sense that whatever is consistent with the basic principles (sooner or later) has a model realization.

spacelike separation results in the commutativity of each admissible intermediate central projector contribution P_{β_1}

$$\begin{aligned} [\psi(x)]_{\beta\beta_1}[\psi(y)]_{\beta_1\beta_2} &= [\psi(y)]_{\beta\beta_1}[\psi(x)]_{\beta_1\beta_2} \\ (x-y)^2 &< 0 \end{aligned} \tag{25}$$

In the setting of analyticity properties of vacuum expectation values the timelike region has cuts in the analytic extensions whereas the spacelike analyticity in Jost points is maintained even after the central decompositions 24 has been carried out. Therefore the formulation in terms of commutation relations requires the presence of projections only for timelike distances whereas for spacelike ones the familiar unprojected form of causality

$$[\psi(x),\psi(y)]=0,\ (x-y)^2<0 \tag{26}$$

is maintained.

The exceptional aspect of two dimensions is caused by the chiral factorization which doubles the degrees of freedom. This peculiar phenomenon has other manifestations. On a classical level the d=1+1 wave equation is the only linear relativistic equation which requires characteristic data on both light cones. In all other cases the avoidance of contradictory overdetermination requires characteristic data on just one lightray/front only. In the local quantum setting of modular holography ([16]) this leads to two different formulae for the wedge algebras

$$\begin{aligned} \mathcal{A}(W) &= \mathcal{A}(R_+)\vee\mathcal{A}(R_-),\ d=1+1 \\ \mathcal{A}(W) &= \mathcal{A}(R_+)=\mathcal{A}(R_-),\ otherwise \end{aligned} \tag{27}$$

Here the $\mathcal{A}(R_\pm)$ are "holographically" associated light-ray/front algebras which are chiral conformal QFT rigorously defined in terms of "modular inclusions" [24] of wedge algebras. The more naive definition of light cone (or p$\rightarrow\infty$) physics in terms of pointlike fields is beset by technical short distance problems which the intrinsic field-coordinate-free algebraic approach avoids. The difference in the two formulas (27) corresponds precisely to the difference in the mentioned classical characteristic value problem for light rays.

Together with the timelike braid group structure one may construct the following: objects a universal observable algebra $\mathcal{A}_{univ}$ on the compactified Minkowski space $\bar{M}$, the charge transporters once around that space in timelike direction and Verlinde's matrix S which is a numerical matrix associated with such a transport. This matrix was discovered by Verlinde [31] in a geometrical reading of QFT (as if chiral theory would lives on a torus) and later Rehren [9] showed that this follows directly without this geometric interpretation from the local quantum physics of charge transports in the presence of spectator charges. The present timelike transport is even further removed from Verlindes framework but for Rehren's algebraic deduction the difference in the interpretation in the present case makes no difference.

The exchange algebra, unlike the canonical (anti)commutation relations is not a "complete" algebraic structure since the distributional aspect at colliding points remains undetermined. More precisely the algebra only fixes the monodromies of the analytically continued correlation functions.

The Rs are obtained by the classification of Markov states on the braid group algebra, a technique which will be sketched in the next section. For the phases which appear in the central decomposition of fields on the covering space (12) we do not need such a detailed knowledge. With the braid group interpretation of the central phases in the block decomposition and their relation to the scaling dimension of fields we have succeeded to generalize the theory of anomalous dimensions from two to higher spacetime dimensions.

Note that most of the concepts and methods used here become meaningless in the euclidean approach. Here again chiral theories has to be treated as an exceptional case because the analytic continuation to the imaginary light ray leads to correlation functions which are as noncommutative as the original real ones; to make contact with classical (commutative) statistical mechanics one has to glue $\pm$ together to obtain d=1+1 Bosons.

The formal relation to braid group structures poses the question whether other characteristic features which one met in the braid group statistics aspects. For example one may ask whether the timelike transport of a charge around $\bar{M}$ in the presence of another spectator charge leads to the numerics which can be arranged into a Verlinde matrix S which together with a diagonal matrix T which is constructed from the central phases forms a SL(2,Z) group. The results in the next section show that this is indeed the case. Furthermore it will also be shown that this modular SL(2,Z) group acts on suitably defined Gibbs states formed with the conformal hamiltonian belonging to the compact time translation τ.

Finally there remains the question of model realizations of these structures i.e. the existence and parametrization of models having a prescribed charge superselection structure. In the conventional approach to d=1+1 and chiral conformal QFT one solves this problem by explicitly constructing i.e. higher level representations of current algebras or by generalizing the Virasoro algebra techniques to W-algebras. This is a cumbersome way which is very far removed from the present braid group structure, but in those few cases where it has been pursued to the end, it does establish existence and uniqueness. Since QFT on a light ray offers no room for coupling strength deformations, chiral QFT is as free fields uniquely determined by its combinatorial superselection data. In the present context it would seem natural to try to formulate a representation theory of exchange algebras, but as mentioned already, this would end soon in an unpleasant surprise: in contrast to free Boson/Fermion (CCR/CAR) algebras the exchange algebras are underdetermined in the sense that the distributional behavior at coalescing points is left open which in the analytic correlation function setting corresponds to the knowledge about monodromies without complete information on the short distance behavior. Though in many cases (including the minimal models) one can determine the 4-point functions by guessing the correct formula for the trajectories of the scale dimension of fields as a function of the "charge quantum numbers" of the sectors they create (by looking at the uniquely determined statistical dimension formula) [21], this is more an artistic than systematic procedure. The recently introduced modular localization formalism [16] promises to furnish a a more systematic approach to this problem.

Higher dimensional conformal theories add to this an additional structure namely the Lorentz covariance of the exchange algebra: it must hold on every timelike line. With other words the problem is not only the study of one chiral exchange algebra on one timelike line but rather a whole family of them living in the same Hilbert space. Closer examination in the algebraic setting reveals that a higher dimensional conformal QFT can be equivalently described by only a finite number of copies of the same chiral algebra which have a special position to each other in a common Hilbert space; in the case at hand this relativ position is determined by finite set of Lorentz-isomorphisms of one abstract chiral algebra in one Hilbert space (with deformation parameters as coupling strength corresponding to different isomorphisms). In fact analogous statements remain even true for massive QFTs[7] where the chiral theories are attached to the "horizons" of a wedge and the finite family forms the basis of what one might call modular holography or scanning [16][25]. A detailed presentation of these interesting developments would go beyond the scope of this article.

Let us briefly collect the obtained results. We have shown that in conformal theories the timelike ordering and Huygens principle allows the appearace of timelike braid group relations between components appearing in the central decomposition at timelike separation. For spacelike separations the sums of the components commute and no central decomposition is needed in order to arrive at spacelike commutation relations.

4 Huygens principle in setting of algebraic QFT

In order to derive the properties of the previous sections which were based on consistency and analogies directly from first principles, we use the methods of algebraic QFT specialized to the conformal case. The most elegant intrinsic and concise way of presenting observable algebras and their representations of physical interest is the superselection theory of Doplicher Haag and Roberts [3]. In that theory one characterizes the observable algebra by a net of spacetime indexed covariant and local von Neumann algebras:

$$\begin{aligned} &\mathcal{O} \rightarrow \mathcal{A}(\mathcal{O}) \\ &net\ \mathcal{A} \equiv \{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{K}} \end{aligned} \tag{28}$$

where $\mathcal{K}$ denotes the Poincaré invariant family (usually taken to be the set of generalized double cones which in the conformal case is left invariant under

[7]The general association of chiral conformal theories with massless QFT is incorrect [16]; only those chiral theories which fulfill more restrictive properties as e.g. possessing an dimension two energy-momentum tensor are massless.

causal complementation and includes wedges) and without loss of generality the $\mathcal{A}(\mathcal{O})$ may be taken as concrete operator (von Neumann closure of C*-) algebras in a common Hilbert space. The advantage of this method is that it is totally intrinsic, i.e. it does not distinguish particular coordinatizations by generating fields (and therefore has a similar relation to the Lagrangian quantization approach as modern differential geometry has to the old coordinate dependent way of doing geometry).

Very fortunately a lot of important physical properties only depend on general causality and spectral aspects which the general principles of local quantum physics impose on these nets of algebras. This is why even for the chiral conformal theories we abstained from the use of the more familiar special algebras as Virasoro- and current- algebras. With the latter one may have an easier start (less conceptual investment), but the problems would begin to show up if one tries to understand higher dimensional conformal QFT along similar lines.

It has been shown that for analysis of charge sectors (fusion, statistics) on local observable algebras such detailed knowledge about specific algebras as mentioned before is not necessary, just a strengthened form of causality called Haag-duality which for chiral theories follows from causality and Moebius-covariance is sufficient. From this one obtains the description of superselection charges in terms of localized endomorphisms of the observable algebra with special properties. The sectors of interests in conformal theories are locally generated i.e. a representation π of the net $\mathcal{A}$ which is globally unitary inequivalent to the vacuum representation π_0 but locally equivalent in the sense

$$\begin{aligned} &\pi(A) \simeq \pi_0(A), \;\; A \in \mathcal{A}(\mathcal{O}') \\ &i.e. \; \pi(\mathcal{A}(\mathcal{O}') = V\pi_0(\mathcal{A}(\mathcal{O}'))V^{-1} \end{aligned} \tag{29}$$

where $\mathcal{O}$ is region from a family $\mathcal{K}$ of regions which is closed under conformal transformations (the smallest natural such family is that of generalized double cones) and the upper dash on a region denotes the causal complement of that region (whereas on an algebra a dash denotes the von Neumann commutant algebra). As the vacuum representation represents the net faithfully, one may (and we will) identify $\mathcal{A}$ with the concrete operator algebra net $\pi_0(\mathcal{A})$ since $\mathcal{A}$ is a faithful representation. Generally representations of algebras allow no natural decomposition as e.g. the tensor composition of group algebras. Therefore it came as somewhat of a surprise that representations of local nets do. One only needs the following strengthened form of causality called Haag duality

$$\mathcal{A}(\mathcal{O}') = \mathcal{A}(\mathcal{O})' \tag{30}$$

(note that by replacing the $=$ by $\subset$ one recovers causality). It is a deep insight that this stronger form, in case where it is not already fulfilled, can be always achieved by an intrinsic extension of the net within the vacuum Hilbert space and that the relation between the original non-dual net and its dual extension contains information about spontaneous symmetry breaking [26]. In conformal theories one expects that spontaneous symmetry breaking cannot occur and therefore the validity of the original unextended duality and in chiral theories one can show that this follows from first principles [27]. By a standard argument this duality property permits to show that the formula

$$\begin{aligned} \rho(A) \quad &: \quad = V^{-1}\pi_0(A)V = AdV^{-1}A,\ A \in \mathcal{A} \\ \pi(A) \quad &= \quad \pi_0 \circ \rho(A) \end{aligned} \tag{31}$$

defines an $\mathcal{O}$-localized endomorphism ($\rho(A) = A,\ A \in \mathcal{A}(\mathcal{O}')$) of the net (the last relation is just a reminder that the net has been identified with its faithful vacuum representation) which acts on the vacuum Hilbert space. It is convenient to define a global "quasilocal" algebra $\mathcal{A}_{quasi}$ as the inductive C*-algebra limit using the fact that double cones in Mikowski space are directed towards infinity and that a directed net of operator algebras has a naturally defined inductive limit. In this way the endomorphisms are endomorphisms of $\mathcal{A}_{quasi}$ with a localization structure. In fact these endomorphisms turn out to be "transportable" i.e. their localization can be arbitrarily changed by charge transporters (unitaries in the algebra $\mathcal{A}_{quasi}$ which possess themselves localization properties) and the superselection sectors are their equivalence classes by unitaries within the net $\mathcal{A}$ [29]. Since endomorphisms can be freely composed, we now have a composition theory of locally generated representation presented on a golden plate

$$\begin{aligned} &\pi_1 \otimes \pi_2 := \pi_0 \circ \rho_1\rho_2 \\ &[\pi_1] \otimes [\pi_2] = [\rho_1\rho_2] = [\rho_2\rho_1] = [\pi_2] \otimes [\pi_1] \end{aligned} \tag{32}$$

where the last relation is between sectors.

In general QFT the composition is only commutative if the localization regions of the two ρ is spacelike separated. The commutativity of sectors is equivalent to the existence of unitary charge transporters $\varepsilon_E(\rho_1, \rho_2)$ with

$$\rho_2\rho_1 = Ad\varepsilon_E(\rho_1, \rho_2) \circ \rho_1\rho_2 \tag{33}$$

For the exchange operator $\varepsilon_E(\rho_1, \rho_2)$ (statistics operator) one obtains an explicit formula by picking two commuting reference endomorphisms and working out the unitary which transports the given situation to the reference one (which may be written in terms of the individual charge transporters and the action of the given endomorphisms on them). One may change the localization of the reference regions and the chosen charge transporters, as long as one keeps the relative distance of the reference configuration spacelike the ε_E will not change.

In the present conformal case we use the Huygens principle for the observable algebra in order to define an analogous timelike localized reference pair of endomorphisms and an associated exchange operator $\varepsilon_H(\rho_1, \rho_2)$. Again as long as we deform this situation maintaining a timelike reference separation the $\varepsilon_H(\rho_1, \rho_2)$ will not change, but the spacelike defined ε is of course a different from the timelike operator. The spacelike exchange operator is as in nonconformal theories the statistics operator because particle/field statistics is related to spacelike exchange. A timelike exchange is one of the more unusual features of conformal QFT in its relation to particle physics. As many such features on should not try to understand it as a physical process but rather look at its consequences which includes the understanding and classification of admissable spectra of scaling dimensions. If one insists in interpreting it as a process in time it would look as (1) in the intoduction.

We introduce a conjugate endomorphism $\bar{\rho}$ to ρ by demanding that $\bar{\rho}\rho$ contains the vacuum sector, i.e. that there exists an intertwiner $R \in (id, \bar{\rho}\rho)$ which induces a standard left inverse ϕ of ρ

$$\phi(A) = R^* \bar{\rho}(A) R \quad \forall\ A \in \mathcal{A} \tag{34}$$

with finite statistics. Here we recall that the left inverse of an endomorphism ρ of $\mathcal{A}$ is a normalized positive linear map satisfying the relation $\phi(\rho(A)B\rho(C)) = A\phi(B)C$. Let us for a moment return to the spacelike exchange situation where this formalism has been known for a long time. There the ϕ has been called regular if it is of the above form, and standard, if in addition the exchange statistics parameter $\lambda_\rho := \phi(\varepsilon(\rho,\rho)) \in \rho(\mathcal{A})'$ is a nonvanishing multiple of a unitary (which then depends only on the sector $[\rho]$). A sufficient condition for the existence of a standard left-inverse and therefore of a conjugate is that there is *some* left-inverse with statistics parameter $\lambda_\rho \neq 0$ ("finite statistics") and that ρ is translation covariant with positive energy condition. The uniqueness of the standard left inverse is a consequence of its definition. Any theory with a mass gap(i.e. a particle interpretation and scattering theory) possesses a standard left inverse [5] and in QFT we should restrict our interest to theories with finite λ. The standard left inverse of ρ turns out to be a trace on $\rho(\mathcal{A})'$. The inverse modulus of λ_ρ is called the *statistical dimension* $d(\rho) \equiv d_\rho \geq 1$. One easily proves that $\lambda_\rho = \lambda_{\bar{\rho}}$ with $\bar{\rho}$ denoting the conjugate endomorphism. For irreducible $\rho' s$ we have $\lambda_\rho = \kappa_\rho d_\rho$ with κ_ρ being the *statistics phase.* One finds one finds in d¿1+1 theories $d_\rho = dim H_\rho$ and $\kappa_\rho = \pm 1$ for Bosons/Fermions [3]. In fact for $d \geq 3+1$ the statistics operator is easily shown on general grounds to fulfill $\varepsilon^2 = 1$ (i.e. absence of monodromies) which leads to permutation group statistics. The statistics structure for low dimensional are much richer because the spacelike exchange leads to braid group statistics and a general spin-statistics theorem. All these things are well-known and there are excellent reviews about them [30].

A moment of thinking reveals that structurally nothing changes in the case of timelike exchange in conformal QFT; we should only remember that the exchange operator ε_H and its numerical consequences is different from the above statistics exchange operator ε_E and we should replace the word "statistics" everywhere by "timelike (or Huygens) exchange" and be aware of the huge difference in physical interpretation and the fact that the braid group is much richer than its special case the permutation group of the spacelike exchange. I appologize if due to accustomization I occasionally forget to keep track of this in my notation..

The above considerations show that for conformal theories this superselection formalism has an novel and interesting extension into the timelike region which is based on Huygens principle i.e. the commutation of timelike separated observables. This Huygens principle related superselection structure maintains is richness even in higher spacetime dimensions (braid group instead of the permutation group for spacelike exchange) where the spacelike situation leads to the mathematically rather barren Boson/Fermion alternative.

An additional concept which shows the interplay of the topological aspect of the conformal covering as well as the resulting global charge structure with the spectrum of anomalous dimensions in a most direct physical way is obtained by the method of global charge transport within an appropiately defined globalized algebra $\mathcal{A}_{univ}$ obtained by a universal construction from the conformal net described in the sequel.

Closely related to the timelike braid group structure in conformal theories is the S-T structure (SL(2,Z)-modular structure) of timelike charge transports around the compactified Minkowski world. A mathematical prerequisite for the setting of global charge transports is the definition of a globalisation of the observable net which is consistent with the compactification of Minkowski space.

The compactification of chiral conformal QFT is most efficiently done in terms of a universal C^*-algebra $\mathcal{A}_{uni}(\bar{M})$ which is different from the noncompact DHR quasilocal algebra $\mathcal{A}_{quasi}(M)$. In order to understands its construction, we note that the net $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\subset\bar{M}}$ is not directed (as the nets of double cones in ordinary Minkowski space) towards infinity. Therefore we should think of a globalization which is different from the standard inductive limit used in the DHR theory. For this we use the following definition universal algebra $\mathcal{A}_{univ}$ [10][28]:

Definition 1 *$\mathcal{A}_{univ}$ is the C^* algebra which is uniquely determined by the system of local algebras $(A(\mathcal{O}))_{\mathcal{O}\in\mathcal{T}}$, $\mathcal{T}$ = family of proper double cones $\mathcal{O}\subset\bar{M}\simeq S^{d-1}\times S^1$ (i.e. their extension in conformal time does not involve all of S^1) and the following universality condition:*

(i) there are unital embeddings $i^I:\mathcal{A}(\mathcal{O})\rightarrow\mathcal{A}_{univ}$ s. t.

$$i^{\mathcal{O}'}\mid_{\mathcal{A}(\mathcal{O})}=i^{\mathcal{O}}\quad if\quad \mathcal{O}\subset\mathcal{O}',\ \mathcal{O},\mathcal{O}'\in\mathcal{T} \tag{35}$$

and $\mathcal{A}_{univ}$ is generated by the algebras $i^I(\mathcal{A}(\mathcal{O}))$, $\mathcal{O}\in\mathcal{T}$;

(ii) for every coherent family of representations $\pi^{\mathcal{O}}:\mathcal{A}(\mathcal{O})\rightarrow\mathcal{B}(H_\pi)$ there is a unique representation π of $\mathcal{A}_{univ}$ in H_π s. t.

$$\pi\circ i^{\mathcal{O}}=:\pi^{\mathcal{O}} \tag{36}$$

The universal algebra inherits the action of the Möbius group as well as the notion of positive energy representation through the embedding.

The universal algebra has more global elements than the quasilocal algebra of the DHR theory: $\mathcal{A}_{quasi} \equiv \mathcal{A} \subset \mathcal{A}_{univ}$ with the consequence that the vacuum representation π_0 ceases to be faithful and the global superselection charge operators which are outer for $\mathcal{A}$ become inner for $\mathcal{A}_{univ}$ as will be shown in the following. From this observation emerges the algebra of Verlinde which originally was obtained by geometric-analytic analogies rather than by local quantum physics arguments. The removal of the compactification i.e. the cutting open along a d-1 submanifold which recreates spacetime infinity of M and also the distinction between past and future light cones as well as the ordering of double cones towards infinity on which the definition of the globalization $\mathcal{A}_{quasi}$ of the net $\mathcal{A}$ was based.

We now study global intertwiners in $\mathcal{A}_{univ}$. Let $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{T}$ and $\xi, \zeta \in \mathcal{O}_1' \cap O_2'$ i.e. two d-1 dimensional subsets whose cuttting out defines the/a d-1 dimensional infinity of Minkowski space and let $\mathcal{A}_\xi, \mathcal{A}_\zeta$ denote the two quasilocal Minkowski space algebras with the net directed towards the ξ, ζ infinity where ξ and η denotes the removal of a point from S^{d-1} and one from S^1 in the product space $S^{d-1} \times S^1$. This amounts to the decompactification creating the a, b $(d-1)$ dimensional cuts in Fig.3. Topologically only the pointlike intersection with the timelike circle S^1 matter in the following argument.

We choose two endomorphisms ρ and σ s. t. $\mathrm{loc}\rho$, $\mathrm{loc}\sigma \subset \mathcal{O}_1$ and $\hat{\rho} \in [\rho]$ with $\mathrm{loc}\hat{\rho} \subset \mathcal{O}_2$ with $\mathcal{O}_1$ and $\mathcal{O}_2$ having no overlap and ξ and ζ lying in the same connected component. Then the local exchange operators (the subscript H omitted) $\varepsilon(\rho, \sigma)$ and $\varepsilon(\sigma, \rho) \in \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}_\xi \cap \mathcal{A}_\zeta$ are the same (i.e. they don't need a label ξ or ζ) independently of whether we use the quasilocal algebra $\mathcal{A}_\xi$ or $\mathcal{A}_\zeta$ for their definition. By Haag duality a charge transporter $V : \pi_0\rho \to \pi_0\hat{\rho}$ lies both in $\pi_0(\mathcal{A}_\xi)$ and $\pi_0(\mathcal{A}_\zeta)$. However its pre-images with respect to the embedding into $\mathcal{A}_{univ}$ are different. In fact:

$$\begin{aligned} V_\rho &\equiv V_+^* V_- \text{ with } V_+ \in \mathcal{A}_\xi, \quad V_- \in \mathcal{A}_\zeta \\ V_\rho &\in (\rho, \rho)_{glob} \end{aligned} \tag{37}$$

is a global selfintertwiner, which is easily shown to be independent of the choice of V and $\hat{\rho}$. The representation of the statistics operators in terms of the charge transporters $\varepsilon(\rho, \sigma) = \sigma(V_+)^* V_+,\ \varepsilon(\sigma, \rho)^* = \sigma(V_-)^* V_-$ leads to:

$$\sigma(V_\rho) = \varepsilon(\rho, \sigma) V_\rho \varepsilon(\sigma, \rho) \ \curvearrowright \pi_0\sigma(V_\rho) = \pi_0 \left[\varepsilon(\rho, \sigma)\varepsilon(\sigma, \rho)\right] \tag{38}$$

The first identity is very different from the relation between $\varepsilon's$ due to local intertwiners. The global intertwiner V_ρ is trivial in the vacuum representation, thus showing its lack of faithfulness with respect to $\mathcal{A}_{univ}$. The global aspect of V_ρ is only activated in charged representations where it coalesces with monodromy operators. From its definition it is clear that it represents a charge transport once around the circular timelike topology of the compactified Minkowski space $\bar{M}$[8].

[8]Note that in A_{univ} which corresponds to a compact quantum world it is not possible to "dump" unwanted charges to "infinity"(as in the case for A_{quasi}), but instead one encounters "polarization" effects upon charge transportation once around.

As a result of its existence, the monodromy around the timelike loop S^1 which is defined as the above two-fold iteration of the braid generator, takes on some of its geometric meaning which it has e.g. in the theory of complex functions. The left hand side of the first equation in (38) expresses a transport "around" in the presence of another charge σ, i.e. a kind of "charge polarization" of ρ in the presence of σ. Let us look at the invariant version of V_ρ namely the global "Casimir" operators $W_\rho = R_\rho^* V_\rho R_\rho : id \to id$. This operator lies in the center $\mathcal{A}_{univ} \cap \mathcal{A}'_{univ}$ and depends only on the class (=sector) $[\rho]$ of ρ. By explicit computation [5] one shows that after the numerical renormalization $C_\rho := d_\rho W_\rho$ one encounters the fusion algebra:

$$\begin{aligned} (i)\ \ C_{\sigma\rho} &= C_\sigma \cdot C_\rho \\ (ii)\ \ C_\rho^* &= C_{\bar{\rho}} \\ (iii)\ \ C_\rho &= \sum_\alpha N^\alpha C_\alpha \ \ if \ \ \rho \simeq \oplus_\alpha N^\alpha \rho_\alpha \end{aligned} \tag{39}$$

Verlinde's modular algebra emerges upon forming matrices with row index equal to the label of the central charge and the column index to that of the sector in which it is measured:

$$S_{\rho\sigma} := \left| \sum_\gamma d_\gamma^2 \right|^{-12} d_\rho d_\sigma \cdot \pi_0 \sigma(W_\rho) \tag{40}$$

In case of nondegeneracy of sectors, which expressed in terms of statistical dimensions and phases means $\left| \sum_\rho \kappa_\rho d_\rho^2 \right|^2 = \sum_\rho d_\rho^2$, the above matrix S is equal to Verlinde's matrix S [31] which together with the diagonal matrix $T = \kappa^{-1} Diag(\kappa_\rho)$, with $\kappa^3 = (\sum_\rho \kappa_\rho d_\rho^2) / \left| \sum_\rho \kappa_\rho d_\rho^2 \right|$ satisfies the modular equations of the genus 1 mapping class group

$$\begin{aligned} SS^\dagger &= 1 = TT^\dagger, \quad TSTST = S \\ S^2 &= C, \quad C_{\rho\sigma} \equiv \delta_{\bar{\rho}\sigma} \\ TC &= CT \end{aligned} \tag{41}$$

It is remarkable that these properties are shared with chiral conformal theories and with d=2+1 plektonic models [9] even though the localization properties of the charge-carrying fields are quite different. In the chiral case one has the additional phase relation:

$$\kappa \left| \kappa \right| = e^{-2\pi i c/8} \tag{42}$$

where c is the constant which measures the strength of the two-point function of the energy-momentum tensor. This relation may be derived by studying the (modular) transformation properties of the Gibbs partition functions for the compact Hamiltonian L_0 of the conformal rotations under thermal duality transformations $\beta \rightarrow 1/\beta$. For d=2+1 plektons, and the present timelike charge transport no physical interpretation is known but the analogy of the timelike spacetime structure with the circular chiral case suggests strongly that the conformal Hamiltonian R_0 should lead to such a thermal duality.

From the central charges Q_ρ and the endomorphisms one may build up an interesting global algebra whose evaluation in the vacuum sector generalizes the numerics of the Verlinde matrix in the direction of higher genus mapping class groups. Whereas the Verlinde matrix S is expected to show up in SL(2,Z) modular properties of Gibbs states which use the R_0 Hamiltonian instead of H (as in chiral conformal field theory) the physical interpretation of the higher ones is not known.

5 Concluding remarks

Many recent investigations of higher dimensional conformal field, in particular perturbative studies of anomalous dimensions have been started from the AdS side. The reason behind this is that the spectrum of anomalous dimensions is given by that of the operator $m_c = \sqrt{R_\mu R^\mu}$, and the only theory which in the setting of functional integrals has an action which is associated with this operator in the standard sense of the classical action-Hamiltonian relation (and which allows to reprocess its data into a conformal QFT) is an AdS field theory. Even more: thanks to the Ads-CQFT isomorphism any consistent AdS QFT will lead to a conformal theory, not just supersymmetric ones or those having vanishing Beta-functions (perturbative AdS meaning?) or contain gravitational aspects. Furthermore, as already remarked before, a Lagrangian input on the AdS side is inevitably causing new (less local) degrees of freedom (in addition to conventional Lagrangian ones) which a Lagrangian framework does not seem to be able to describe. So one could ask the question if string theory with its latest surprising achievement as a search machine for geometrical aspects of particle problems does not also indicate its demise as a framework for conceptual competence concerning local quantum physical aspects of particle physics.

In our structure analysis of spectra of anomalous dimensions we did nor use the AdS side. This of course does not mean that it would be futile to translate the present findings into the AdS side. To the contrary this could be very interesting and one might expect that many aspects which were conceptually clear on the conformal side but for which the standard formalism does not supply a good algorithm will show the exact opposite behavior on the AdS side.

The most fascinating but at the same time very speculative idea which emerges from the present results on the structure of higher dimensional conformal theories is the suggestion that the inner symmetry situation in nature with the conspicuous absence of exact nonabelian group symmetries but some leftover regularities, may be the remnant of the timelike plektonic structure. The idea is that the multiplicities which can be generated by the timelike plektonic structure may be responsible for the regularities in the massive theory. In view of our poor understanding of natural processes which may drive conformal theories towards massive ones, this idea remains vague. But it is worthwile to stress the fact that interacting conformal quantum field theories do not only violate the prerequisites of the Coleman-Mandula theorem, but they enter that forbidden terrain of inexorable inner-spacetime symmetry rather deeply in a way which without this Huygens exchange mechanism would have been unimaginable.

I believe that it would be very fruitful to study more particle physics consequences of the present results by extending the existing work on particle-like excitations [11] and their generalized scattering theory [8] without and/or with the AdS setting. Although such suggestions about next steps are usually rendered obsolete within a short time by unfolding events, they may be useful to get the particle physics theory ball rolling again; even if they cannot control its direction.

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A Classification of admissable B_∞ representations

The charge-carrying fields form an exchange algebra (called the "reduced field bundle") in which R-matrices which represent the infinite braid group B_∞ appear. The admissable physical representations define a so called Markov trace on the braidgroup, a concept which was introduced by V. Jones but already had been used for the special case of the permutation group S_∞ in the famous 1971 work of Doplicher, Haag and Roberts [3]. Since this very physical method has remained largely unknown[9], we use its present interest in connection with the Huygens sector structure of higher dimensional conformal theories as a "bait" for its popularization.

In this classification approach one starts with fusion channels of endomorphisms. The simplest case is a basic endomorphism with a two channel fusion

$$\rho^2 \simeq id \oplus \rho_1 \tag{43}$$

$$i.e. \left[\rho^2\right] = [id] \oplus [\rho_1] \tag{44}$$

where id is the identity endomorphism. This is the famous case leading to the

[9] Particle physicists who are very familiar with group theory use a deformation theory known as the "quantum group" method. Although its final results are compatible with the structure of quantum theory, the intermediate steps are not (no Hilbert space&operator algebras, appearance of null-ideals). The present method is quantum all the way.

Jones-Temperley-Lieb algebra, whereas the more general two-channel case

$$\rho^2 \simeq \rho_1 \oplus \rho_2 \tag{45}$$

gives rise to the Hecke algebra. Finally the special 3-channel fusion

$$\rho^2 \simeq id \oplus \rho_1 \oplus \rho_2 \tag{46}$$

yield the so called Bierman-Wenzl algebra. Each single case together with the Markov trace yields of a wealth of braid group representations. The first case comprises all the selfconjugate minimal models which in quantum group language are obtained by deforming $SU(2)$ (a pseudo self-conjugate group), whereas the second covers the quantum deformations of $SU(n)$ for n¿2. Finally the third one belongs to the SO(n) deformations. There are of course also isolated exceptional fusion laws which do not produce families and whose basic fusion law cannot be viewed as arising from loking at closed subsets of higher composites from the above. In all such cases one finds a "quantization" from the positivity of the Markov-trace; in the first case this is the famous Jones quantization, the second and third case has a current algebra as well as a W-algebra realization. The classification of the admissable braid group representation associated to the above fusion laws (and the associated knot- and 3-manifold- invariants) is a purely combinatorial problem of which a simpler permutation group version (for which only (45) occurs) was already solved in 1971 by DHR. The method requires to study tracial states on the mentioned abstract C*-algebras and the resulting concrete von Neumann algebras are factors of type II_1. These operator algebras which are too "small" in order to be able to carry even continuous translations (afortiori no localizations) and are often referred to as "topological field theories". Unlike quantum mechanics their relation to Feynman Kac representability remains "artistic" (i.e. nonintrinsic). This means that their functional integral derivation in terms of Chern-Simons actions using the Witten prescriptions is a one way street; there is no mathematical theory which leads back from the noncommutative operator algebras to any sort of euclidean Feynman-Kac representation[10]. These combinatorial data are part of the superselection data. If combined with the nature of the charge-carrying fields i.e. the information wether they form multiplets as in the case of current algebras or whether there are no such group theoretic multiplicities the have the same R-matrices and the same statistical dimensions (quantum dimensions) but their statistical phases and therefore their anomalous dimensions may be different. The numerical R-matrices determined from the Markov trace formalism fix the structure of the exchange algebras.

The DHR-Wenzl technique constructs the tracial states via iterated applicarion of the left inverse of endomorphisms. Under the assumption of irreducibility

[10] In principle in the absence of noncanonical short distance properties (see quantum mechanics or the ϕ_2^4 theory) the Feynman-Kac measure may be reconstructed from the analytically continued euclidean correlations.

of ρ (always assumed in the rest of this section) ϕ maps the commutant of $\rho^2(\mathcal{A})$ in $\mathcal{A}$ into the complex numbers:

$$\phi(A) = \varphi(A)\underline{1}, \quad A \in \rho^2(\mathcal{A})' \tag{47}$$

and by iteration a faithful tracial state φ on $\cup_n \rho^n(\mathcal{A})'$ with:

$$\begin{aligned} \phi^n(A) &= \varphi(A)\underline{1}, \quad A \in \rho^{n+1}(\mathcal{A})' \\ \varphi(AB) &= \varphi(BA), \quad \varphi(\underline{1}) = 1 \end{aligned}$$

Restricted to the $\mathbf{C}RB_n$ algebra generated by the ribbon braid-group which is a subalgebra of $\rho^n(\mathcal{A})'$ the φ becomes a tracial state, which can be naturally extended ($B_n \subset B_{n+1}$) to $\mathbf{C}RB_\infty$ in the above manner and fulfills the "Markov-property":

$$\varphi(a\sigma_{n+1}) = \lambda_\rho \varphi(a), \quad a \in \mathbf{C}RB_n \tag{48}$$

The terminology is that of V. Jones and refers to the famous russian probabilist of the last century as well as to his son, who constructed knot invariants from suitable functionals on the braid group. The "ribbon" aspect refers to an additional generator τ_i which represents the vertical 2π rotation of the cylinder braid group ($\simeq$ projective representation of B_n) [5].

It is interesting to note that the Markov-property is the combinatorial relict of the cluster property which relates the n-point correlation function in local QFT to the n-1 point correlation or in QM the physics of n particles to that of n-1 (rendering one particle a spectator by removing it to infinity. The infinite permutation- and braid groups are the only groups behaving like a russian "matrushka" i.e. the smaller ones are naturally contained in the bigger. This picture is similar to that of cluster properties which was already used in our attempts to understand statics in the nonrelativistic setting of the first chapter. The existence of a Markov trace on the ribbon braid group of (low dimensional) multi-particle statistics is the imprint of the cluster property on particle statistics. As such it is more basic than the notion of internal symmetry. It precedes the latter and according to the DR theory it may be viewed as the other side of the same coin on which one side is the old (compact group-) or new (quantum-) symmetry. With these remarks the notion of internal symmetry becomes significantly demystified.

Let us now return to the above 2-channel situation. Clearly the exchange operator ε_ρ has maximally two different eigenprojectors since otherwise there would be more than two irreducible components of ρ^2. On the other hand ε_ρ cannot be a multiple of the identity because ρ^2 is not irreducible. Therefore ε_ρ has exactly two different eigenvalues λ_1, λ_2 i.e.

$$(\varepsilon_\rho - \lambda_1\underline{1})(\varepsilon_\rho - \lambda_2\underline{1}) = 0 \tag{49}$$

$$\leftrightarrow \varepsilon_\rho = \lambda_1 E_1 + \lambda_2 E_2 \ , \quad E_i = (\lambda_i - \lambda_j)^{-1} (\varepsilon_\rho - \lambda_j) \, , \quad i \neq j \tag{50}$$

which after the trivial re-normalization of the unitaries $g_k := -\lambda_2^{-1}\rho^{k-1}(\varepsilon_\rho)$ yields the generators of the Hecke algebra:

$$\begin{aligned} g_k g_{k+1} g_k &= g_{k+1} g_k g_{k+1} \\ g_k g_l &= g_l g_k \ , \quad |j-k| \geq 2 \\ g_k^2 &= (t-1) g_k + t \ , \quad t = -\lambda_1 \lambda_2 \neq -1 \end{aligned} \tag{51}$$

The physical cluster property in the algebraic form of the existence of a tracial Markov state leads to a very interesting "quantization" [11]. Consider the sequence of projectors:

$$E_i^{(n)} := E_i \wedge \rho(E_i) \wedge ... \wedge \rho^{n-2}(E_i) \ , \quad i = 1, 2 \tag{52}$$

and the symbol $\wedge$ denotes the projection on the intersection of the corresponding subspaces. The notation is reminiscent of the totally antisymmetric spaces in the case of Fermions. The above relation $g_1 g_2 g_1 = g_2 g_1 g_2$ and $g_1 g_n = g_n g_1$, $n \geq 2$ in terms of the E_i reads:

$$\begin{aligned} E_i \rho(E_i) E_i - \tau E_i &= \rho(E_i) E_i \rho(E_i) - \tau \rho(E_i) \ , \quad \tau = t\,(1+t)^2 \\ E_i \rho^n(E_i) &= \rho^n(E_i) E_i \ , \quad n \geq 2 \end{aligned} \tag{53}$$

The derivation of these equations from the Hecke algebra structure is straightforward. The following recursion relation [32] of which a specialisation already appeared in [3] is however tricky and will be given in the sequel

Proposition 2 *The projectors $E_i^{(n)}$ fulfill the following recursion relation ($t = e^{2\pi i\alpha}$, $-\pi 2 < \alpha < \pi 2$) :*

$$\begin{aligned} E_i^{(n+1)} &= \rho(E_i^{(n)}) - 2\cos\alpha \sin n\alpha \sin(n+1)\alpha \rho(E_i^{(n)}) E_j \rho(E_i^{(n)}) \ , \quad i \neq j, \quad n+1 \leq q \\ E_i^{(q)} &= \rho(E_i^{(q-1)}) \quad , \quad q = \inf\{n \in \mathbf{N}, n\,|\alpha| \geq \pi\} \ \ if \ \alpha \neq 0, \ q = \infty \ if \ \alpha = 0 \end{aligned} \tag{54}$$

The DHR recursion for the permutation group S_∞ is obtained for the special case t=0 i.e. $\alpha = 0$. In this case the numerical factor in front of product of three operators is $nn+1$.

The proof is by induction. For n=1 the relation reduces to the completeness relation between the two spectral projetors of ε_ρ : $E_i = 1 - E_j$, $i \neq j$. For the induction we introduce the abbreviation $F = E_j \rho(E_i^{(n)}) = \rho(E_i^{(n)}) E_j$ and

[11] In these notes we use this concept always in the original meaning of Planck as a discretization, and not in the modern form of a deformation.

compute F^2. We replace the first factor $\rho(E_i^{(n)})$ according to the induction hypothesis by:

$$\rho(E_i^{(n)}) = \rho^2(E_i^{(n-1)}) - 2\cos\alpha\sin(n-1)\alpha\sin n\alpha\rho^2(E_i^{(n-1)})\rho(E_j)\rho^2(E_i^{(n-1)}) \tag{55}$$

We use that the projector $\rho^2(E_i^{(n-1)})$ commutes with the algebra $\rho^2(\mathcal{A})'$ (and therefore with $E_j \in \rho^{(2)}(\mathcal{A})'$), and that its range contains that of $\rho(E_i^{(n)})$ i.e. $\rho^2(E_i^{(n-1)})\rho(E_i^{(n)}) = \rho(E_i^{(n)})$. Hence we find:

$$F^2 = E_j\rho(E_i^{(n)}) - 2\cos\alpha\sin(n-1)\alpha\sin n\alpha\rho^2(E_i^{(n-1)})E_j\rho(E_j)E_j\rho(E_i^{(n)}) \tag{56}$$

Application of (53) with $\tau = 12\cos\alpha$ to the right-hand side yields:

$$F^2 = E_j\rho(E_i^{(n)}) - \sin(n-1)\alpha 2\cos\alpha\sin\alpha\rho^2(E_i^{(n-1)})E_j\rho(E_i^{(n)}) = \sin(n+1)\alpha 2\cos\alpha\sin n\alpha F \tag{57}$$

where we used again the above range property and a trigonometric identity. For $n = q-1$ the positivity of the numerical factor fails and by $F^2E_j = (FF^*)^2$ and $FE_j = FF^*$ the operator F must vanish and hence E_j is orthogonal to $\rho(E_j^{(q-1)})$ which is the second relation in (54). For $n < q-1$ the right-hand side of the first relation in (54) with the help of (57) turns out to be a projector which vanishes after multiplication from the right with $\rho^k(E_j), k = 1, ..., n-2$ as well as with E_j. The remaining argument uses the fact that this projector is the largest with this orthogonality property and therefore equal to $E_i^{(n+1)}$ by definition of $E_i^{(n+1)}$ q.e.d.

The recursion relation (54) leads to the desired quantization after application of the left inverse ϕ :

$$\begin{aligned} \phi(E_i^{(n+1)} &= E_i^{(n)}\left(1 - 2\cos\alpha\sin n\alpha\sin(n+1)\alpha\eta_j\right), \quad i \neq j \\ \eta_j &= \phi(E_j),\ 0 \leq \eta_j \leq 1,\ \eta_1 + \eta_2 = 1 \end{aligned} \tag{58}$$

¿From this formula one immediately recovers the permutation group DHR quantization for $\alpha = 0$. In that case positivity of the bracket restricts η_j to the values $12(1 \pm 1d),\ d \in \mathbf{N} \cup 0$. For $\alpha \neq 0$ one first notes that from the second equation (54) one obtains (application of ϕ):

$$\eta_j E_i^{(q-1)} = \phi(E_j\rho(E_i^{(q-1)})) = \phi(E_jE_i^{(q)}) = 0, \quad i \neq j \tag{59}$$

where the vanishing results from the orthogonality of the projectors. Since $\eta_1 + \eta_2 = 1$ we must have $E_i^{(q-1)} = 0$ for i=1,2, q≥ 4, because $E_i^{(q-1)} \neq 0$ would imply $\eta_j = 0$ and $E_j^{(q-1)} = 0$. This in turn leads to $E_j \equiv E_j^{(2)} = 0$ which contradicts the assumption that ε_ρ processes two different eigenvalues. This is obvious for $q = 3$ and follows for $q > 3$ from the positivity of ϕ (58) for n=2:

$$\phi(E_j^{(3)}) = -\sin\alpha \sin 3\alpha E_j^{(2)} \quad \curvearrowright E_j^{(2)} = 0 \quad \curvearrowright E_i^{(q-1)} = 0,\ i = 1, 2,\ q \geq 4 \tag{60}$$

Using (58) iteratively in order to descend in n starting from $n = q-2$, positivity demands that there exists an $k_i \in \mathbf{N}, 2 \leq k_i \leq q-2$, with:

$$\eta_i = \sin(k_i + 1)\alpha 2 \cos\alpha \sin k_i\alpha,\ i = 1, 2 \quad \curvearrowright \sin(k_1 + k_2)\alpha = 0 \tag{61}$$

where the relation results from summation over i. Since the only solutions are $\alpha = \pm\pi q$, $k_1 = d$, $k_2 = q - d$, $d \in N$, $2 \leq d \leq q-2$, one finds for the statics parameters of the plektonic 2-channel family the value:

$$\lambda_\rho = \sum_{i=1}^{2} \lambda_i \eta_i = -\lambda_2 \left[(t+1)\,\eta_1 - 1\right] = -\lambda_2 e^{\pm\pi i(d+1)/q} \sin\pi/q \sin d\pi/q \tag{62}$$

a formula which allows for a nice graphical representation. We have established the following theorem:

Theorem 3 *Let ρ be an irreducible localized endmorphism such that ρ^2 has exactly two irreducible subrepresentations. Then:*

- ε_ρ has two different eigenvalues λ_1, λ_2 with ratio

$$\lambda_1 \lambda_2 = -e^{\pm 2\pi i/q}, \quad q \in \mathbf{N} \cup \{\infty\},\ q \geq 4 \tag{63}$$

- The modulus of the statistics parameter $\lambda_\rho = \phi(\varepsilon_\rho)$ has the possible values

$$, d \in N,\ \ 2 \leq d \leq q - 2(64)$$

$$|\lambda_\rho| = \begin{cases} \sin\pi/q \sin d\pi/q, & q < \infty \\ 1d, 0 & q = \infty \end{cases} \quad , d \in N,\ \ 2 \leq d \leq q-2$$

The representation $\varepsilon_\rho^{(n)}$ of the braid group B_n which is generated by $\rho^{(k-1)}(\varepsilon_\rho)$, $k = 1, ..., n-1$ in the vacuum Hilbert space is an infinite multiple of the Ocneanu-Wenzl representation tensored with a one dimensional (abelian) representation. The projectors $E_2^{(m)}$ and $E_1^{(m)}$ are "cutoff" (vanish) for $d < m \leq n$ and $q - d < m \leq n$ resp.

The iterated left inverse $\varphi = \phi^n$ defines a Markov trace tr on B_n :

$$tr(b) = \varphi \circ \varepsilon_\rho(b) \tag{65}$$

The "elementary" representation which is characterized by two numbers d and q gives rise to a host of composite representation which appear if one fuses the ρ, ρ_1, ρ_2 and reduces etc. We will not present the associated composite braid formalism. With the same method one can determine the statistical phases up to an anyonic (abelian) phase. In order to have a unique determination, one needs (as in the original DHR work) an additional piece of information which e.g. may consist in specifying the lowest power of ρ which contains the identity endomorphism (the vacuum representation) for the first time. A special case of this is $\rho^2 \supset id$ i.e. the selfconjugate Jones-Temperley-Lieb fusion. Here we will not present these computations of phases.

The problem of 3-channel braid group statistics has also been solved with the projector method in case that one of the resulting channels is an automorphism τ:

$$\rho^2 = \rho_1 \oplus \rho_2 \oplus \tau \tag{66}$$

In that case ε_ρ has 3 eigenvalues λ_i which we assume to be different:

$$(\varepsilon_\rho - \lambda_1)(\varepsilon_\rho - \lambda_2)(\varepsilon_\rho - \lambda_3) = 0 \tag{67}$$

The relation to the statistics phases ω_ρ, ω_i is the following: $\mu_i^2 = \omega_i \omega^2$. In addition to the previous operators $G_i = \rho^{i-1}(\varepsilon_\rho) = (G_i^{-1})^*$ we define projectors:

$$E_i = \rho^{i-1}(TT^*)$$

where $T \in (\rho^2 \,|\tau)$ is an isometry and hence E_i the projector onto the eigenvalue $\lambda_3 = \lambda_\tau$ of G_i. In fact one finds the following relations between the G_i and E_i :

$$\begin{aligned} E_i &= \mu_3(\mu_3 - \mu_1)(\mu_3 - \mu_2)(G_i - (\mu_1 + \mu_2) + \mu_1\mu_2 G_i^{-1}) \\ E_i G_i &= \mu_3 E_i \end{aligned} \tag{68}$$

This together with the trilinear relations between the $G_i's$ and $E_i's$ as well as the commutation of neighbors with distance ≥ 2 gives upon renormalization the operators g_i and e_i which fulfill the defining relation of the Birman-Wenzl algebra which again depends on two parameters. The Markov tracial state classification again leads to a quantization of these parameters except for a continuous one-parameter solution with statistical dimension $d = 2$ which is realized in conformal QFT as sectors on the fixed point algebra of the $U(1)$ current algebra (which has a continuous one-parameter solution) under the action of the charge conjugation transformation (often called "orbifolds" by analogy to constructionsw in differential geometry).

Finally one may ask the question to what extend these families and their descendends+ some known isolated exeptional cases exhaust the possibilities of plektonic exchange structures. Although there are some arguments in favor, the only rigorous mathematical statement is that of Rehren who proved that for exchange dimension $d < \sqrt{6}$ that this is indeed the case [33]

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